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DYNAMIC CONTACT PROBLEMS FOR AN ORTHOTROPIC ELASTIC HALF-PLANE AND A COMPOSITE PLANE*

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The equivalence properties of certain boundary-value problems are studied for ortho- and isotropic elastic domains in stationary subsonic dynamics and statics. By using these properties, solutions obtained earlier in explicit form for dynamic contact problems for an isotropic half-plane with four, and a composite half-plane with six, kinds of boundary conditions /1/ are extended to the case of an orthotropic material whose characteristic equation has pure imaginary roots. An analogous problem is solved for an arbitrary orthotropic half-plane by constructing new complex potentials. The existence and uniqueness of the real root of the Rayleigh equation governing the rate of surface wave propagation in an orthotropic half-plane with a free boundary are proved. The problem of the breaking of a totally adherent stamp from an orthotropic half-plane is solved in terms of elementary functions. The existence of a single delamination section is proved its length is found, and the contact stresses are calculated.

Savin /2/ and Galin /3/ first examined static mixed problems for an anisotropic half-plane with the simultaneous formulation of two kinds of boundary conditions. The corresponding dynamic stationary problems, with the exception of problems with total adhesion conditions, can be solved by uing Barenblatt-Cherepanov potentials /4/.

1. Let x_1, y be rectangular Cartesian coordinates for which the directions of the axes coincide with the principal elasticity directions of an orthotropic plane, $x = x_1 - ct$, y is a coordinate system moving at constant velocity c with respect to this plane, and t is the time.

We define stationary deformations of the plane in the x, y system by equilibrium equations, the generalized Hooke's law, and the condition of continuity

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=\alpha\frac{\partial^2 u}{\partial x^2},\quad \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=\alpha\frac{\partial^2 v}{\partial x^2},\quad \alpha=\rho c^2 \tag{1.1}$$

$$\varepsilon_x=\beta_{11}\sigma_x+\beta_{12}\sigma_y,\quad \varepsilon_y=\beta_{12}\sigma_x+\beta_{22}\sigma_y,\quad \varepsilon_{xy}=\beta_{66}\tau_{xy} \tag{1.2}$$

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y},\quad \varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{1.3}$$

$$\beta_{kk}>0,\quad \beta_0^2-\beta_{12}^2>0,\quad \beta_0=\sqrt{\beta_{11}\beta_{22}}$$

where β_{jk} are orthotropy coefficients and ρ is the density of the material.

Combining the derivatives with respect to x and using the last equalities in (1.2) and (1.3), we can write (1.1) in a form without inertial terms

$$\partial\sigma_x^*/\partial x+\partial\tau_{xy}/\partial y=0,\quad \partial\tau_{xy}/\partial x+\partial\sigma_y^*/\partial y=0 \tag{1.4}$$
$$\sigma_x^*=\sigma_x-\alpha u',\quad \sigma_y^*=(1-\alpha\beta_{66})^{-1}(\sigma_y+\alpha u'),\quad u'=\partial u/\partial x$$

Substituting (1.4) into (1.2) we obtain Hooke's law for the new "orthotropic" plane with "stresses" σ_x^*, σ_y^*, τ_{xy}

$$\varepsilon_x=\gamma_{11}\sigma_x^*+\gamma_{12}\sigma_y^*,\quad \varepsilon_y=\gamma_{21}\sigma_x^*+\gamma_{22}\sigma_x^*,\quad \varepsilon_{xy}=\gamma_{66}\tau_{xy} \tag{1.5}$$
$$\gamma_{11}=\beta\beta_{11},\quad \gamma_{12}=\beta\beta_{12}(1-\alpha\beta_{66}),\quad \gamma_{21}=\beta[\beta_{12}-\alpha(\beta_0^2-\beta_{12}^2)]$$
$$\gamma_{22}=\beta(1-\alpha\beta_{66})[\beta_{22}-\alpha(\beta_0^2-\beta_{12}^2)],\quad \beta=[1-\alpha(\beta_{11}-\beta_{12})]^{-1},$$
$$\gamma_{66}=\beta_{66}$$

An analogous relation holds in the spatial problem.

Eqs.(1.3)-(1.5) agree, in form, with the statics equations for the orthotropic plane (1.1)-(1.3) with $c=0$. Consequently, a static problem for the same domain with the previous conditions for the displacements and shear stresses corresponds to a boundary-value problem

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for an orthotropic domain in a dynamic formulation but with different orthotropy coefficients which depend on the inertial parameter α and other kinds of boundary conditions in place of conditions on the stresses σ_x, σ_y governed by (1.4). In this sense the problems examined are equivalent and the same methods are applicable to them if they do not rely on the symmetry of Hooke's law because $\gamma_{12} \neq \gamma_{21}$. The method of complex potentials belongs to such methods: its underlying Eq.(1.3) is common for (1.1)-(1.3) and (1.3)-(1.5).

2. Let us use the Lekhnitskii potentials in terms of which the solution of the "static" problem (1.3)-(1.5) is expressed in the form /5/

$$\sigma_x^* = \operatorname{Re} \sum_{k=1}^{2} \mu_k^2 \varphi_k(z_k), \quad \sigma_y^* = \operatorname{Re} \sum_{k=1}^{2} \varphi_k(z_k), \quad \tau_{yx} = -\operatorname{Re} \sum_{k=1}^{2} \mu_k \varphi_k(z_k) \tag{2.1}$$

$$u' = \operatorname{Re} \sum_{k=1}^{2} p_k \varphi_k(z_k), \quad v' \equiv \frac{\partial v}{\partial x} = \operatorname{Re} \sum_{k=1}^{2} s_k \varphi_k(z_k)$$

$$p_k = \gamma_{11}\mu_k^2 + \gamma_{12}, \quad s_k = \gamma_{21}\mu_k + \gamma_{22}\mu_k^{-1}, \quad z_k = x + \mu_k y$$

Here $\varphi_k(z_k)$ are functions analytic in the z_k plane, and μ_k are the roots of the equation

$$\gamma_{11}\mu^4 + (\gamma_{12} + \gamma_{21} + \gamma_{66})\mu^2 + \gamma_{22} = 0 \tag{2.2}$$

such that $\mu_1 \neq \mu_2$, $\operatorname{Im} \mu_k > 0$. Solutions for multiple and real roots are not examined. They correspond to cases of an exceptional combination of orthotropy parameters (isotropy in particular) and losses of the ellipticity of the system of elasticity theory equations, which requires a change in the form of the solution (2.1).

Changing to stresses in the original dynamic problem (1.1)-(1.3), we obtain according to (1.4)

$$\sigma_x = \operatorname{Re} \sum_{k=1}^{2} [\mu_k^2 + \alpha(\gamma_{12} + \mu_k^2\gamma_{11})] \varphi_k(z_k) \tag{2.3}$$

$$\sigma_y = \operatorname{Re} \sum_{k=1}^{2} r_k \varphi_k(z_k), \quad r_k = 1 - \alpha(\gamma_{12} - \gamma_{66} + \mu_k^2\gamma_{11})$$

The remaining components retain their form (2.1).

3. The solution of (2.1)-(2.3) loses meaning formally for a real root μ_k because the plane z_k degenerates into the line $y = 0$. In substance this occurs because, being analytic, it describes only fairly smooth deformation fields characteristic for subsonic regimes. In the isotropic plane the boundary of such regimes is determined by the shear wave velocity $c_2 = (\rho\beta_{66})^{-1/2}$. In an anisotropic material the magnitude of the wave velocity depends on its direction; consequently, the ellipticity boundary of the system of elasticity equations for this material cannot be expressed in terms of some wave velocity by a universal formula. From physical considerations it is conceivable that the ellipticity upper boundary might be the least c^* from the plane longitudinal dilatation wave velocity c_1 and shear wave velocity c_2.

Since a plane dilatation wave is symmetric while a shear wave is skew-symmetric about the x-axis, they can be considered as "surface" waves in the orthotropic half-plane $y < 0$, respectively, for slipping and antislipping support of the boundary $y = 0$. Applying a bilateral Laplace transformation in x to (1.1)-(1.3), we obtain the characteristic functions $R_k = R_k(c) = \sqrt{1 - c^2 c_k^{-2}}$, $k = 1, 2$, whose zeros have the form $c_1 = \sqrt{\beta_{22}}\,[\rho(\beta_0^2 - \beta_{12}^2)]^{-1/2}$, $c_2 = (\rho\beta_{66})^{-1/2}$ from the conditions 1) $\tau_{xy} = v' = 0$ and 2) $\sigma_y = u' = 0$ for $y = 0$. Analysis of (2.2) confirms that because of the change in sign of the quantity $\gamma_{11}\gamma_{22}$ from plus to minus $(\gamma_{22} = \beta\beta_{22}R_1^2(c) R_2^2(c))$ when c passes through $c^* = \min(c_1, c_2)$, at least one of the roots μ_k will become real for $c > c^*$, and ellipticity is lost. Examples of orthotropy can be mentioned when this would even occur for $c < c^*$. Later just the "subsonic" regime $c \in [0, c^*)$ will be considered.

We similarly obtain the characteristic functions (and Rayleigh equation)

$$R_3(c) = R_2(c)(1 - \alpha\beta_{11}) - \alpha\beta_0 R_1(c) = 0$$

$$R_4(c) = \sqrt{\beta_{11}}\, c_2^{-2} R_1(c) + \sqrt{\beta_{22}}\, c_1^{-2} R_2(c) = 0$$

from the conditions 3) $\tau_{xy} = \sigma_y = 0$ and 4) $u = \dot{v} = 0$.

The real zeros of these functions, if they exist and do not generate real roots of (2.2), determine the surface wave velocity in the free and clamped half-planes; the functions $R_k(c)$ themselves occur in the solution of mixed problems for an orthotropic half-plane (see Sects. 4 and 6).

4. We will investigate the zeros of the function $R_3(c)$ for $c \in (0, c^*)$. If $c_2 < c_1$, then $c^* = c_2$, $R_3(c^*) = -\rho c_2^2 \beta_0 R_1(c_2) < 0$; if $c_2 > c_1$, then $c^* = c_1$, $R_3(c^*) = \beta_{12}^2(\beta_{12}^2 - \beta_0^2)$ $R_2(c_1) < 0$; if $c_2 = c_1$, then the function $R_3(c) = [1 - \alpha(\beta_{11} + \beta_0)] R_1(c)$ is equivalent to an infinitely small $AR_1(c)$ as $c \to c^* - 0$, where $A = (\beta_{12}^2 + \beta_{22}\beta_0)(\beta_{12}^2 - \beta_0^2)^{-1} < 0$, and is therefore negative. In all cases $R_3(0) = 1$, and therefore, the zero $c = c_R$ of the function $R_3(c)$ exists in the interval $(0, c^*)$.

To prove its uniqueness it is sufficient to prove that $R'(\alpha_*) < 0$, where $R(\alpha) \equiv R_3(c)$, $\alpha_* = \rho c_R^2$.

Differentiating with respect to α and taking into account that $R(\alpha_*) = 0$ we obtain

$$R'(\alpha_*) = -\{\beta_{11}R_2^* + \beta_0[R_1^* + {}^1/_2\rho(c_1^2 - c_2^2)c_R^2(R_1^*R_2^*)^{-1}]\}, \quad R_k^* = R_k(c_R)$$

therefore $R'(\alpha_*) < 0$ for $c_1 \geqslant c_2$.

Let $c_1 < c_2$. Since $R(\alpha_*) = 0$, we have $1 - \beta_{11}\alpha_* > 0$,

$$\alpha_* \in (0, \beta_{11}^{-1}), \quad R^+(\alpha) \equiv R_2(c)(1 - \alpha\beta_{11}) + \alpha R_1(c) \beta_0 > 0$$

for $\alpha \in (0, \beta_{11}^{-1})$, the zeros of the functions $P(\alpha) \equiv R^+(\alpha) R(\alpha) = (1 - \alpha\beta_{11})^2 R_2^2(c) - \alpha^2\beta_{11}R_1^2(c)$ and $R(\alpha)$ coincide in $(0, \beta_{11}^{-1})$; if $P'(\alpha_*) < 0$, then $R'(\alpha_*) < 0$ also.

We will assume the opposite: $P'(\alpha_*) \geqslant 0$. We introduce the quantities $\varepsilon = \beta_{66}\alpha_*$ and $\varkappa = c_2^2 c_1^{-2}$ that vary in the domain $0 < \varepsilon < 1$, $\varkappa > 1$, $\varepsilon\varkappa < 1$. It follows from the condition $P(\alpha_*) = 0$ that $\beta_0^2 = (1 - \varepsilon)(\beta_{66} - \beta_{11})^2 (1 - \varepsilon\varkappa)^{-1}\varepsilon^{-2}$. Hence and from the condition $P'(\alpha_*) \geqslant 0$ we obtain

$$(1 - \beta_{11}\alpha_*)\left\{\left(1 - \frac{1-\varepsilon}{1-\varepsilon\varkappa}\right)\beta_{11} - \left[\frac{(1-\varepsilon)\varkappa}{1-\varepsilon\varkappa} - \frac{2}{\varepsilon} + 1\right]\beta_{66}\right\} \geqslant 0$$

or by using the estimates $1 - \beta_{11}\alpha_* > 0$, $1 - \varkappa < 0$, we obtain the inequality

$$\beta_{11} \leqslant B\beta_{66}, \quad B = (2 - \varepsilon - 3\varepsilon\varkappa + 2\varepsilon^2\varkappa)(1 - \varkappa)^{-1}\varepsilon^{-2} \tag{4.1}$$

that is satisfied only for $B > 0$. Let $B > 0$, $\beta_0^2 = C\beta_{66}^2$, then $C > 0$, $B\beta_{22} > C\beta_{66}$. Taking into account that $c_2^2 = \varkappa c_1^2$, we hence have $C(B - \varkappa)\beta_{66}^2 \geqslant B\beta_{12}^2$ or $B \geqslant \varkappa$ or $B = \varkappa N$, where $N \geqslant 1$. By virtue of (4.1) the function

$$f(\varepsilon, \varkappa, N) = 2 - \varepsilon - 3\varepsilon\varkappa + 2\varepsilon^2\varkappa - \varepsilon^2\varkappa(1 - \varkappa)N$$

should have a zero in the domain $\varepsilon \in (0, 1)$, $\varkappa > 1$, $N \geqslant 1$.

We introduce the function $f_1(\varepsilon, \varkappa) \equiv f(\varepsilon, \varkappa, 1)$. In the new variables $\lambda = 1 - \varepsilon$, $\mu = \varkappa - 1$ we obtain

$$f_1(\varepsilon, \varkappa) = (\lambda - \mu)(2\lambda - \mu) + \lambda\mu(3\lambda - 2\mu + \lambda\mu), \quad \lambda \in (0, 1), \quad \mu > 0$$

It follows from the condition $\varepsilon\varkappa < 1$ that $\lambda\mu > \lambda - \mu$. Setting $\lambda\mu = \lambda - \mu + t$, $t > 0$, we obtain $f_1(\varepsilon, \varkappa) = t(\lambda + t)$. Since $f_1(\varepsilon, \varkappa) > 0$ for all $\varepsilon, \varkappa, N$, the function $f(\varepsilon, \varkappa, N)$ is positive where it should have a zero. This contradiction shows that $R'(\alpha_*) < 0$.

Therefore, a real root of the equation $R_3(c) = 0$ exists and is unique in the interval $(0, c^*)$. The result obtained is a solution of the problem formulated in /4/. However, the question as to whether a wave exists that propagates in an orthotropic half-plane at the Rayleigh velocity $c_R \in (0, c^*)$ for any elastic characteristics β_{jk}, can be answered affirmatively only by proving that (2.2) has no real root for $c = c_R$. Such a proof has been obtained and will be published separately.

According to (3.1) there are no zeros of the function $R_4(c)$ in the interval $(0, c^*)$ therefore, stationary waves do not exist in a clamped orthotropic half-plane, including the isotropy case /1/.

5. If the roots of (2.2) are purely imaginary $\mu_k = i\nu_k$, $\nu_1 \neq \nu_2$ then still another analogy exists between the stationary dynamics problems (1.1)-(1.3) for orthotropic and isotropic domains. To clarify it we turn to (1.3)-(1.5) which we solve in the new form

$$\begin{gathered} A_1\sigma_y = \operatorname{Re}[q\varphi_1(z_1) + q_2\varphi_2(z_2)], \quad A_2\tau_{xy} = \operatorname{Im}[q_1\varphi_1(z_1) + q\varphi_2(z_2)] \\ A_3u' = -\operatorname{Re}[\varphi_1(z_1) + q_2\varphi_2(z_2)], \quad A_4v' = \operatorname{Im}[q_1\varphi_1(z_1) + \varphi_2(z_2)]; \\ z_k = x + i\nu_k y \end{gathered} \tag{5.1}$$

where $q, q_1, q_2, A_1, \ldots, A_4$ are real numbers. Equating multipliers of the functions $\varphi_k(z_k)$ in the four corresponding expressions of (5.1) and (2.1), (2.3), we obtain successively

$$\begin{gathered} A_3 = -p_1^{-1}, \quad A_4 = -is_2^{-1}, \quad q_1 = s_1s_2^{-1}, \quad A_2 = s_1(s_2\nu_1)^{-1} \\ q = s_1\nu_2(s_2\nu_1)^{-1}, \quad A_1 = qr_1^{-1}, \quad q_2 = A_1r_2 \end{gathered} \tag{5.2}$$

Since these seven coefficients (real only for imaginary roots of (2.2)) must satisfy eight conditions, an additional constraint arises

$$p_1s_1\nu_2r_2 = p_2s_2\nu_1r_1 \tag{5.3}$$

It can be shown that it is always satisfied.

Indeed, using the notation

$$P = p_1 s_1 \nu_2 r_2 - p_2 s_2 \nu_1 r_1, \quad \chi_k = p_k + \gamma_{66} \tag{5.4}$$

we obtain $r_k = 1 - \alpha\chi_k$, $s_k = \nu_k\chi_k$ from (2.3) and (2.1) by virtue of (2.2). Hence and from (5.4) it follows that

$$P = \nu_1\nu_2 (p_1 - p_2)(\gamma_{66} + p_1 + p_2 - \alpha\chi_1\chi_2)$$

Taking into account that $p_k = \gamma_{12} - \nu_k^2\gamma_{11}$ and according to (2.2) by the Viete formulas $\nu_1^2\nu_2^2 = \gamma_{22}\gamma_{11}^{-1}$, $\nu_1^2 + \nu_2^2 = \gamma_{11}^{-1}(\gamma_{12} + \gamma_{21} + \gamma_{66})$, we obtain

$$P = \nu_1\nu_2 (p_1 - p_2)[\gamma_{12} - \gamma_{21} - \alpha(\nu_{11}\gamma_{22} - \gamma_{12}\gamma_{21} - \gamma_{66}\gamma_{21})] \equiv 0 \tag{5.5}$$

since the content of the square brackets vanishes identically on substituting expressions (1.5) in place of γ_{jk}.

Apart from unimportant factors A_k, the solution (5.1) and (5.2) is identical in form with the solutions of stationary dynamic problems for isotropic half-planes and a composite plane /1/. Hilbert-Riemann boundary-value problems corresponding to mixed conditions of four and six kinds that govern the contact of the half-plane boundary and the composite plane edges with different stamps and cover plates are formulated and solved in /1/. In order to write down the solution of the same problem, say, for an orthotropic half-plane, the functions $u' = u_0(x)$, $v' = v_0(x)$, etc. given for $y = 0$ in /1/ must be replaced by the functions $A_3u_0(x)$, $A_4v_0(x), \ldots,$ the coefficients $A_1, A_2, \ldots, q_2$ must be calculated by means of (5.2) and we must put $\mu = 1$. In conformity with (5.1), the quantities A_3u', A_4v', etc. will be components of the solution obtained; unlike /1/ they do not degenerate for $c = 0$ (excluding the isotropy case, of course) and therefore include the static problem.

6. Let the roots of (2.2) be complex. We change to the form (2.1), (2.3) and construc a solution of the problem for an orthotropic half-plane $y < 0$ on sections $L_1, \ldots, L_4$ for which the boundary ($y = 0$) contact conditions ($L_j \cap L_k = 0$ for $j \neq k$) are posed

$$u_1' \equiv u' = u_0(x), \quad x \in L_2 \cup L_3; \quad u_2' = v' = v_0(x), \quad x \in L_1 \cup L_3 \tag{6.1}$$

$$\sigma_{12} \equiv \tau_{xy} = \tau_0(x), \quad x \in L_1 \cup L_4; \quad \sigma_{22} \equiv \sigma_y = \sigma_0(x), \quad x \in L_2 \cup L_4$$

We note that by partitioning the boundary conditions into symmetric and skew-symmetric ones, the problem for a homogeneous orthotropic plane weakened by a system of rectilinear slits on whose opposite edges and outside the slits identical conditions of nine kinds are posed in the form of the given functions

$$1)\ \sigma_y, \tau_{xy}, \quad 2)\ u, v, \quad 3)\ v, \tau_{xy}, \quad 4)\ u, \sigma_y, \quad 5)\ v, [u], [\tau_{xy}],$$
$$6)\ \sigma_y, [u], [\tau_{xy}], \ 7)\ \tau_{xy}, [v], [\sigma_y], \ 8)\ u, [v], [\sigma_y], \ 9)\ [u], [v], [\sigma_y], [\tau_{xy}]$$

is reduced to two such problems.

They govern the contact between the edges with inextensible flexible stringers and completely ahderent or slipping rigid stamps, as well as their mutual comb-like adhesion and contact through the slipping or welded stringers embedded in the slit; $[f]$ is the jump in the function $f(x)$.

Following /1/, we set

$$\tfrac{1}{2}\varphi_k(z) = A_{k1}\Phi(z) + A_{k2}\overline{\Phi}(z), \quad \overline{\Phi}(z) = \overline{\Phi(\bar{z})}, \quad z = x + iy \tag{6.2}$$

where A_{kl} are arbitrary complex constants, $\Phi(z)$ is a piecewise analytic function for which the conditions of the problem a jump occur on L_4 and conditions of the Riemann problem on L_3.

If the boundary values of the functions (6.1) are sought in the form

$$(-1)^k\sigma_{k2} = 2\operatorname{Re}[\alpha_k(\Phi^+(x) - \Phi^-(x))], \quad x \in L_4, \quad k = 1, 2 \tag{6.3}$$

$$-u_k' = 2\operatorname{Re}[\alpha_{k1}\Phi^+(x) - \alpha_{k2}\Phi^-(x)], \quad \alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}, \quad x \in L_3 \tag{6.4}$$

where α_k and α_{kl} are complex numbers, then these conditions are satisfied for $\operatorname{Im} S \neq 0$ and $\operatorname{Im} \bar{N} \neq 0$ because we have by virtue of (6.3) and (6.4)

$$\sigma_{22} - \bar{S}\sigma_{12} = -\alpha_1(S - \bar{S})(\Phi^+(x) - \Phi^-(x)), \quad S = -\alpha_2\alpha_1^{-1}, \quad x \in L_4 \tag{6.5}$$

$$u_1' + \bar{N}u_2' = (\alpha_{11} + \bar{N}\alpha_{21})\Phi^+(x) - (\alpha_{12} + \bar{N}\alpha_{22})\Phi^-(x), \quad x \in L_3 \tag{6.6}$$

$$N = -\alpha_{11}\alpha_{21}^{-1} = -\alpha_{12}\alpha_{22}^{-1}$$

Let us calculate the coefficients in (6.2)-(6.6). We substitute (6.2) into (2.3). Equating factors for the functions $\Phi^+(x)$ and $\Phi^-(x)$ in (2.3) and (6.3), we obtain

$$\bar\mu_1 A_{12} + \bar\mu_2 A_{22} = -\mu_1 A_{11} - \mu_2 A_{21} = \alpha_1$$
$$\bar r_1 A_{12} + \bar r_2 A_{22} = -r_1 A_{11} - r_2 A_{21} = \alpha_2$$

We set $\alpha_1 = -1$, then $\alpha_2 = S$.
Solving this system, we obtain

$$A_{11} = \Delta_2(S)\Delta,\ A_{12} = -\bar\Delta_2(S)\bar\Delta,\ A_{21} = -\Delta_1(S)\Delta,\ A_{22} = \bar\Delta_1(S)\bar\Delta \tag{6.7}$$
$$\Delta^{-1} = \mu_1 r_2 - \mu_2 r_1 = (\mu_1 - \mu_2) R_2 R_1 \beta,\quad \Delta_k(S) = r_k + \mu_k S,\ \bar\Delta_k(S) = \overline{\Delta_k(\bar S)}$$

Substituting (6.7) into (6.2) and (6.2) into (2.1), it follows from the condition that the multipliers for $\Phi^+(x)$ and $\Phi^-(x)$ are equal in the corresponding components (2.1) and (6.4) that

$$\alpha_{11} = \bar P_1 + S\bar P_2,\ \alpha_{12} = P_1 + SP_2,\ \alpha_{21} = \bar Q_1 + S\bar Q_2 \tag{6.8}$$
$$\alpha_{22} = Q_1 + SQ_2,\ P_1 = (p_1 r_2 - p_2 r_1)\Delta,\ P_2 = (p_1\mu_2 - p_2\mu_1)\Delta$$
$$Q_1 = (q_1 r_2 - q_2 r_1)\Delta,\quad Q_2 = (q_1\mu_2 - q_2\mu_1)\Delta$$

The connection (6.6) between α_{kj} and N, and formulas (6.8) generate the quadratic equations

$$S^2 \operatorname{Im}(P_2\bar Q_2) + S\operatorname{Im}(P_1\bar Q_2 + \bar P_2 Q_1) + \operatorname{Im}(P_1\bar Q_1) = 0 \tag{6.9}$$
$$N^2 \operatorname{Im}(Q_1\bar Q_2) + N\operatorname{Im}(P_1\bar Q_2 + \bar P_2 Q_1) + \operatorname{Im}(P_1\bar P_2) = 0$$

The roots of these equations are purely imaginary.
Let us show this. Substituting p_k, r_k, s_k from (2.1), (2.3) and (6.8), we obtain

$$P_1 = i\mu_0\gamma_{11}R_2^2\Delta^*,\ Q_1 = -P_2 = (\gamma_0 + \gamma_{12})\Delta^*,\ Q_2 = i\mu_0\gamma_0\Delta^* \tag{6.10}$$
$$\Delta^* = [1 - \alpha(\gamma_0 + \gamma_{12} + \gamma_{66})]^{-1},\ i\mu_0 = \mu_1 + \mu_2,\ \gamma_0 = \sqrt{\gamma_{11}\gamma_{22}}$$

Since $\operatorname{Re}(\mu_1 + \mu_2) = 0,\ c < c^*,\ \beta_{11} > 0$, the numbers $\mu_0 > 0,\ \gamma_0,\ \Delta^*,\ P_2,\ Q_1$ are real, the numbers $P_1,\ Q_2$ are imaginary, $\operatorname{Im}(P_1\bar Q_2) = \operatorname{Im}(\bar P_2 Q_1) = 0$. Hence, from (6.9) and (6.10) and taking account of the relationships (5.5), (6.4), (6.8), we have

$$S = -N = i\gamma_{11}^{1/4}\gamma_{22}^{-1/4}R_2(c)$$

Therefore, S and N are imaginary numbers for $c \in [0, c^*]$ according to (1.5).
The boundary values of the functions (6.3) and (6.4) take the form

$$E_0 u' = -2s_0 \operatorname{Im}(Q^+\Phi^+ - Q^-\Phi^-),\ \sigma_y = -2s_0 \operatorname{Im}(\Phi^+ - \Phi^-) \tag{6.11}$$
$$E_0 v' = -2\operatorname{Re}(Q^+\Phi^+ - Q^-\Phi^-),\ \tau_{xy} = 2\operatorname{Re}(\Phi^+ - \Phi^-),$$
$$E_0 \equiv E_0(c) = R_2(c) R_3(c)$$
$$s_0 = -iS = \beta_{11}^{1/4}\beta_{22}^{-1/4}R_1^{-1/2}(c)R_2^{1/2}(c),\quad Q^\pm \equiv Q^\pm(c) = [\gamma_0(1 \pm s_0\mu_0) + \gamma_{12}]\beta^{-1}$$

Substituting (6.11) into (6.1) and taking account of (6.5) and (6.6) we obtain the combined Hilbert-Riemann problem solved in /1/ for a plane with the slits $L_1 \cup L_2 \cup L_3 \cup L_4$

$$\operatorname{Re}\Phi^\pm(x) = -1/2\,(Q^+ - Q^-)^{-1}[E_0 v_0(x) + Q^\mp\tau_0(x)],\quad x \in L_1 \tag{6.12}$$
$$\operatorname{Im}\Phi^\pm(x) = -1/2\, s_0^{-1}(Q^+ - Q^-)^{-1}[E_0 u_0(x) - Q^\mp\sigma_0(x)],\ x \in L_2$$
$$\Phi^+(x) - Q\Phi^-(x) = -1/2 E_0 (s_0 Q^+)^{-1}[s_0 v_0(x) + iu_0(x)],\quad Q = Q^-/Q^+,\ x \in L_3$$
$$\Phi^+(x) - \Phi^-(x) = 1/2\,[\tau_0(x) - is_0^{-1}\sigma_0(x)],\quad x \in L_4$$

Formulas (6.11) show that when the signs of the boundary stresses are preserved the signs of the boundary displacements, as for an isotropic half-plane, are opposite in the sub- and super-Rayleigh ranges $(0, c_R)$ and (c_R, c^*). Since $E_0(c_R) = 0$, then $iu' + s_0 v' \neq 0$, $\sigma_y = \tau_{xy} = 0$, for $c = c_R$ on the boundary $y = 0$, which denotes the existence of a Rayleigh wave.
We will investigate the behaviour of the singularities at points of boundary condition separation during passage to the super-Rayleigh velocity. To this end, we will examine the

fundamental problem for an orthotropic half-plane $y<0$ with the boundary conditions $u' = \delta(x)$, $v' = 0$ for $y = 0$, where $\delta(x)$ is the Dirac function. Solving the Dirichlet problem we obtain directly from (2.1) and (2.3)

$$\tau_{xy}(x, 0) = \rho\mu_0\gamma_0 [\pi\beta \sqrt{\beta_{22}} R_2(c) R_4(c) x]^{-1} \qquad (6.13)$$

The same problem in the form (2.1), (2.3) and (6.2)-(6.11) has the solution

$$\tau_{xy}(x, 0) = -E_0\mu_0\gamma_0 [\pi\beta Q^+Q^- x]^{-1} \qquad (6.14)$$

Comparing (6.13) and (6.14), we obtain $Q^+Q^- = -\rho^{-1}\sqrt{\beta_{22}} R_2^2 R_3 R_4$. Since $R_2^2(c) R_4(c) \neq 0$ for $c \in (0, c^*)$, according to Sect.4, the function $Q^+(c) Q^-(c)$ together with $R_3(c)$ has just one simple zero $c = c_R$ in $(0, c^*)$. Since the functions $Q^\pm(c)$ are bounded in $(0, c^*)$, it follows that the function $Q(c)$, which is a coefficient in the Riemann conditions, changes sign only when c passes through c_R. In $[0, c_R)$ the function $Q(c)$ is continuous in β_{ik} and c, and $Q(c) < 0$ for $c = 0$ for an isotropic material. Consequently, even in the general orthotropy case $Q(c) < 0$ for $c \in [0, c_R)$.

In view of the change in the sign of $Q(c)$ from minus to plus, according to (6.12), the usual root behaviour of the stresses under the edges of completely adherent isolated stamps vanishes for $c \in [0, c_R)$ at super-Rayleigh velocities by conserving oscillations; the stresses under stamps are also bounded in the class of solutions having a bounded elastic strain energy, including even the isotropy case. The root singularities in the stresses under the edges of slipping stamps and stringers remain irrespective of whether they have common points with the totally adherent stamps or not /6/.

7. We examine the problem of the breaking off of an infinite stamp from a totally adherent orthotropic half-plane moving at a sub-Rayleigh velocity by a concentrated force $(P_1, -P_2)$ applied to the free half-plane boundary at a distance a from the end of a crack $x = y = 0$ moving at the same velocity. We shall assume that the crack is closed in the section $x \in [-b, 0)$, $y = 0$ and that the contact coefficient of friction is zero, $v(x) \leqslant 0$ for $x \in [-a, -b]$, and the intersection of the stamp and high-plane behind the force for $x \in (-\infty, -a)$ exerts a weak influence on the solution for $x > -a$ and cannot be taken into account.

According to (6.1) and (6.12), the boundary conditions of this problem

$$\tau_{xy} = P_1\delta(x+a),\ \sigma_y = -P_2\delta(x+a),\ x \in (-\infty, -b) \qquad (7.1)$$
$$v' = \tau_{xy} = 0,\ x \in [-b, 0);\ u' = v' = 0,\ x \in [0, \infty),\ a > b$$

and the solution (2.1), (2.3) (6.2) generate a combined Dirichlet-Riemann problem ($\theta = 0$ for $P_1 = 0$, $P_2 > 0$)

$$\Phi^+ - \Phi^- = g(x),\ x \in (-\infty, -b) \qquad (7.2)$$
$$\Phi^+ = Q\Phi^-,\ x \in [0, \infty);\ \operatorname{Re}\Phi^\pm = 0,\ x \in [-b, 0)$$
$$g(x) = {}^1/_2 iTe^{-i\theta}\delta(x+a),\ iTe^{i\theta} = P_1 + is_0^{-1}P_2,\ |\theta| \leqslant \pi$$

Using the method of Sect.1 in /6/, the canonical solution of problem (7.2) is obtained in the form

$$X(z) = \frac{e^{i\varphi(z)}}{\sqrt{z}},\ \varphi(z) = -2\gamma \ln\frac{1}{i}\left(\sqrt{\frac{z}{b}} + \sqrt{1+\frac{z}{b}}\right),\ \gamma = -\frac{1}{2\pi}\ln(-Q) \qquad (7.3)$$
$$0 \leqslant \arg z \leqslant 2\pi,\quad 0 \leqslant \arg(z+b) \leqslant 2\pi$$
$$\varphi^\pm(x) = \psi_1(x),\ \psi_1(x) = -2\gamma\ln\left(\sqrt{-b^{-1}x-1} + \sqrt{-b^{-1}x}\right),$$
$$x \in (-\infty, -b)$$
$$\varphi^\pm(x) = \pm i[\pi\gamma - \psi_2(x)],\ \psi_2(x) = 2\gamma \operatorname{arctg}\sqrt{-x(b-x)^{-1}},$$
$$x \in (-b, 0)$$
$$\varphi^\pm(x) = \pm i\pi\gamma + \psi_3(x),\ \psi_3(x) = -2\gamma\ln\left(\sqrt{b^{-1}x+1} + \sqrt{b^{-1}x}\right),$$
$$x \in [0, \infty)$$
$$X(z) \sim e^{-\pi\gamma}({}^1/_4 b)^{i\gamma} z^{-1/2-i\gamma},\quad z \to \infty$$

For $z = b$ the function $X(z)$ is bounded and for $z = 0$ has a root singularity $\gamma > 0$, since $0 < -Q < 1$. Setting

$$\Phi(z) = X(z)F(z),\quad F(z) = F_0(z) + F_1(z) \qquad (7.4)$$
$$F_0(z) = \frac{1}{2\pi i}\int_{-\infty}^{-b} \frac{g(t)\,dt}{X^+(t)(t-z)} = -\frac{Te^{-i\theta}}{4\pi(z+a)X^+(-a)} =$$

$$-iT\sqrt{a}\exp[-i(\psi_a+\theta)][4\pi(z+a)]^{-1},\quad \psi_a=\psi_1(-a)$$

we obtain a Dirichlet boundary-value problem from (7.2) for a plane with a slit $\operatorname{Im} F_1^{\pm}(x) = -\operatorname{Im} F_0(x),\ x\in[-b,0)$, where the desired function $F_1(z)$ should be constant as $z\to\infty$ bounded at $z=0$ and integrable for $z=-b$. This function has the form /7/

$$F_1(z)=C_0+iD_0Y_0(z)-\frac{Y_0(z)}{\pi}\int_{-b}^{0}\frac{\operatorname{Im}F_0(t)\,dt}{Y_0^{+}(t)(t-z)} \tag{7.5}$$

$$Y_0(z)=z^{1/2}(z+b)^{-1/2},\quad Y_0^{\pm}(x)=\pm i(-x)^{1/2}(x+b)^{-1/2}$$

where C_0 and D_0 are arbitrary real constants.

Evaluating the integral /8/

$$\int_{-b}^{0}\frac{dt}{(t+a)Y_0^{+}(t)(t-z)}=\frac{\pi i}{z+a}\left[\frac{1}{Y_0(z)}-\frac{1}{Y_0(-a)}\right],\quad Y_0(-a)=\sqrt{\frac{a}{a-b}}$$

we finally obtain

$$\Phi(z)=X(z)[C_0+iD_0Y_0(z)+T_1(z+a)^{-1}+iT_2Y_0^{-1}(-a)(z+a)^{-1}Y_0(z)] \tag{7.6}$$

$$iT_1+T_2=-(4\pi)^{-1}T\sqrt{a}e^{i(\psi_a+\theta)}$$

$$\Phi(z)\sim e^{-\pi\gamma}(^1/_4b)^{i\gamma}z^{-1/2-i\gamma}[C_0+iD_0+O(z^{-1})],\quad z\to\infty$$

Let there be no root field at infinity. Then $C_0=D_0=0$.

$$\Phi(z)=-\frac{Te^{i\varphi(z)}}{4\pi(z+a)}\left[\sin(\psi_a+\theta)\sqrt{\frac{a}{z}}+i\cos(\psi_a+\theta)\sqrt{\frac{a-b}{z+b}}\right] \tag{7.7}$$

Smooth adherence of a stamp at the point $z=-b$ is ensured when the condition $\sigma_y(-b)=0$, or, according to (6.11) and (7.7), the condition $\cos(\psi_a+\theta)=0$ is satisfied. Therefore

$$\Phi(z)=-(4\pi)^{-1}T\sqrt{a}\sin(\psi_a+\theta)e^{i\varphi(z)}(z+a)^{-1/2}z^{-1/2} \tag{7.8}$$

The stresses

$$\sigma_y=-2s_0\operatorname{Im}(\Phi^{+}-\Phi^{-})=\frac{T}{\pi}s_0\sqrt{a}\sin(\psi_a+\theta)\frac{\operatorname{sh}[\pi\gamma-\psi_2(x)]}{(x+a)\sqrt{-x}} \tag{7.9}$$

on the slip section $[-b,0)$ will be compressive for $\psi_a+\theta=-^1/_2\pi+2k\pi$, because $0<\psi_2(x)\leqslant\pi\gamma$, $\cos(\psi_a+\theta)=0$. It hence follows that the quantity b can have just a denumerable set of values

$$\theta-2\gamma\ln(\sqrt{ab^{-1}-1}+\sqrt{ab^{-1}})=-^1/_2\pi+2k\pi,\quad k=0,\ \pm1,2,\ldots \tag{7.10}$$

According to the theorem on the existence and uniqueness of the solution of a contact problem with unilateral constraints /9/, not more than one value of k, determined by the condition of non-intersection $v(x)\leqslant0$ for $x\in(-a,-b)$ can correspond to any contact mode. Since $v(-b)=0$, this condition is satisfied if, in particular, $v'(x)\geqslant0$ in $(-a_1,-b]$.

We will show that actually $v'(x)\geqslant0$. Using (7.8) and (7.3), we have in $[-a,-b]$

$$v'(x)=-T[2\pi E_0(x+a)]^{-1}\sqrt{-ax^{-1}}(Q^{+}-Q^{-})\sin\psi_1(x) \tag{7.11}$$

$$\psi_1'(x)=\gamma(x^2+bx)^{-1/2}>0,\quad x\in[-a,-b]$$

By virtue of (7.3) and (7.11), the function $\psi_1(x)$ is continuous and increases monotonically in $(-a_1,-b]$, $v(-b)=0$, therefore $\psi_a\leqslant\psi_1(x)\leqslant0$, or taking (7.10) into account, we obtain $-\theta-{}^1/_2\pi+2k\pi\leqslant\psi_1(x)\leqslant0$. Since $Q^{+}>Q^{-}$, hence and from (7.11) under the condition $v'(x)\geqslant0$ we obtain $-\pi\leqslant\psi_1(x)\leqslant0$. It follows from these inequalities, written in the form $-\pi\leqslant 2k\pi-\theta_1-{}^1/_2\pi\leqslant0$, and the constraint $|\theta|\leqslant\pi$, that $k=0$, $-^1/_2\pi\leqslant\theta\leqslant{}^1/_2\pi$. This last inequality means that the solution naturally does not exist for $P_2<0$.

The length of the slip section is determined from (7.10) for $k=0$:

$$b = a\,\mathrm{ch}^{-2}\,[{}^1/_2\,({}^1/_2\pi + \theta)\,\gamma^{-1}]$$

Its largest and smallest values are attained for $\theta = \mp{}^1/_2\pi$. In the former case $b = a$ and the slip goes to the point $x = -a$, while in the latter case intersection of the stamp and half-plane starts at once behind this point, which is in agreement with the Cerruti solution.

Let us calculate the stresses and their asymptotic forms on the continuation of the moving crack. Taking into account that $\sin(\psi_a + \theta) = -1$, for $k = 0$ in (7.8) (as also in (7.9)), we obtain for $x > 0$

$$\Phi^{\pm}(x) = \pm T\sqrt{a}\,[4\pi(x + a)\sqrt{x}]^{-1}\exp[\mp\pi\gamma + i\psi_3(x)]$$

Hence, and from (6.11), it follows that

$$\tau_{xy} - is_0^{-1}\sigma_y = 2(\Phi^+ - \Phi^-) = -T\sqrt{a}\,\mathrm{ch}\,\pi\gamma\,[\pi(x + a)\sqrt{x}]^{-1}\exp[i\psi_3(x)]$$

$$\tau_{xy}(x) \sim T\,\mathrm{ch}\,\pi\gamma\,(\pi\sqrt{ax})^{-1},\quad \sigma_y(x) = O(\sqrt{x}),\quad x \to +0$$

Therefore, the crack propagates only because of shear fracture, where the sign of the shearing force P_1 itself has no influence on the stress intensity factor according to (7.2).

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